5 Linear transformations

Linear Transformations Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$ for us).

- A **linear transformation** from $\mathcal{U}$ into $\mathcal{V}$ is defined to be a linear function T mapping $\mathcal{U}$ into $\mathcal{V}$. That is,

  $$T(\boldsymbol{x}+\boldsymbol{y}) = T(\boldsymbol{x})+T(\boldsymbol{y}) \quad \text{and} \quad T(\alpha\boldsymbol{x}) = \alpha T(\boldsymbol{x})$$

  or, equivalently,

  $$T(\alpha\boldsymbol{x}+\boldsymbol{y}) = \alpha T(\boldsymbol{x}) + T(\boldsymbol{y})$$

  for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}$, $\alpha \in \mathbb{F}$.
- A linear operator on $\mathcal{U}$ is defined to be a linear transformation from $\mathcal{U}$ into itself – i.e., a linear function mapping $\mathcal{U}$ back into $\mathcal{U}$.

1. Determine which of the following functions are linear operators on $\mathbb{R}^2$. (a) $T(x,y) = (x, 1+y)$, (b) $T(x,y) = (y,x)$, (c) $T(x,y) = (0, xy)$, (d) $T(x,y) = (x^2, y^2)$, (e) $T(x,y) = (x, \sin y)$, (f) $T(x,y) = (x+y, x-y)$.

2. For $A, X \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, determine which of the following functions are linear transformations. (a) $T(X) = AX - XA$, (b) $T(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ for $\boldsymbol{b} \neq 0$, (c) $T(A) = A^\top$, (d) $T(X) = (X + X^\top)/2$.

3. Explain why $T(\mathbf{0}) = \mathbf{0}$ for every linear transformation T.

4. Determine which of the following mappings are linear operators on $\mathcal{P}_n$, the vector space of polynomials of degree n or less.

(a) $T = \xi_k D^k + \xi_{k-1} D^{k-1} + ... + \xi_1 D + \xi_0 I$, where D^k is the k^{th}-order differentiation operator (i.e., $D^k p(t) = d^k p/dt^k$).

(b) $T\Big(p(t)\Big) = t^n p'(0) + t$.

5. Let $\boldsymbol{v}$ be a fixed vector in $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}$ be the mapping defined by $T(\boldsymbol{x}) = \boldsymbol{v}^\top \boldsymbol{x}$ (i.e., the standard inner product).

(a) Is T a linear operator?

(b) Is T a linear transformation?

Coordinates of a Vector Let $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ be a basis for a vector space $\mathcal{U}$, and let $\boldsymbol{v} \in \mathcal{U}$. The coefficients α_i in the expansion $\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}2 + ... + \alpha_n \boldsymbol{u}_n$ are called the coordinates of $\boldsymbol{v}$ with respect to $\mathcal{B}$, and, from now on, $[v]_\mathcal{B}$ will denote the column vector

$$[\boldsymbol{v}]_\mathcal{B} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Caution! Order is important. If $\mathcal{B}'$ is a permutation of $\mathcal{B}$, then $[\boldsymbol{v}]_{\mathcal{B}'}$ is the corresponding permutation of $[\boldsymbol{v}]_\mathcal{B}$.

6. If $\boldsymbol{v}$ is a vector in $\mathbb{R}^3$ whose standard coordinates are $\boldsymbol{v} = \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix}$. determine the coordinates of $\boldsymbol{v}$ with respect to the basis

$$\mathcal{B} = \left\{ \boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},\ \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

7. First four Lagrange[5] polynomials are 1, $1-t$, $2 - 4t + t^2$ are $6 - 18t + 9t^2 - t^3$. Show that the set that contains these four polynomials is a basis for vector space $\mathcal{P}_3$ ($\mathcal{P}_3$ is the set of all polynomials of degree less or equal to 3). Find a polynomial $q \in \mathcal{P}_3$ such that $[q]_\mathcal{B} = (-2, 0, 1, 0)^\top$ if $\mathcal{B}$ is a basis from above (a basis that contains four Lagrange polynomials).

Space of Linear Transformations

- For each pair of vector spaces $\mathcal{U}$ and $\mathcal{V}$ over $\mathbb{F}$, the set $\mathcal{L}(\mathcal{U}, \mathcal{V})$ of all linear transformations from $\mathcal{U}$ to $\mathcal{V}$ is a vector space over $\mathbb{F}$.
- Let $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ and $\mathcal{B}' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively, and let B_{ji} be the linear transformation from $\mathcal{U}$ into $\mathcal{V}$ defined by $B_{ji}(\boldsymbol{u}) = \xi_j \boldsymbol{v}_i$, where $(\xi_1, \xi_2, ..., \xi_n)^\top = [\boldsymbol{u}]_\mathcal{B}$. That is, pick off the j^{th} coordinate of $\boldsymbol{u}$, and attach it to $\boldsymbol{v}_i$.
  - ▷ $\mathcal{B}_\mathcal{L} = \{B_{ij}\}_{i=1,\dots,m}^{j=1,\dots,n}$ is a basis for $\mathcal{L}(\mathcal{U}, \mathcal{V})$.
  - ▷ $\dim\mathcal{L}(\mathcal{U}, \mathcal{V}) = (\dim\mathcal{U})(\dim\mathcal{V})$.

[5] Joseph Louis Lagrange (1736–1813), born in Turin, Italy, is considered by many to be one of the two greatest mathematicians of the eighteenth century - Euler is the other. Lagrange occupied Euler's vacated position in 1766 in Berlin at the court of Frederick the Great who wrote that "the greatest king in Europ" wishes to have at his court "the greatest mathematician of Europe." After 20 years, Lagrange left Berlin and eventually moved to France. Lagrange's mathematical contributions are extremely wide and deep, but he had a particularly strong influence on the way mathematical research evolved. He was the first of the top-class mathematicians to recognize the weaknesses in the foundations of calculus, and he was among the first to attempt a rigorous development.

Coordinate Matrix Representations Let $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ and $\mathcal{B}' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m\}$ be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. The coordinate matrix of $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ with respect to the pair $(\mathcal{B}, \mathcal{B}')$ is defined to be the $m \times n$ matrix

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [T(\boldsymbol{u}_1)]_{\mathcal{B}'} & [T(\boldsymbol{u}_2)]_{\mathcal{B}'} & \dots & [T(\boldsymbol{u}_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}.$$

In other words, if $T(\boldsymbol{u}_j) = \alpha_{1j}\boldsymbol{v}_1 + \alpha_{2j}\boldsymbol{v}_2 + ... + \alpha_{mj}\boldsymbol{v}_m$, then

$$[T(\boldsymbol{u}_j)]_{\mathcal{B}'} = \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} \quad \text{and} \quad [T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix}.$$

When T is a linear operator on $\mathcal{U}$, and when there is only one basis involved, $[T]_{\mathcal{B}}$ is used in place of $[T]_{\mathcal{B}\mathcal{B}}$ to denote the (necessarily square) coordinate matrix of T with respect to $\mathcal{B}$.

8. Consider the claim Space of Linear Transformations from above. (i) Prove that $\mathcal{B}_{\mathcal{L}} = \{B_{ij}\}_{i=1,...,m}^{j=1,...,n}$ is a basis for $\mathcal{L}(\mathcal{U}, \mathcal{V})$. (ii) Show that $\dim\mathcal{L}(\mathcal{U}, \mathcal{V}) = (\dim\mathcal{U})(\dim\mathcal{V})$.

9. If P is the projector that maps each point $\boldsymbol{v} = (x, y, z) \in \mathbb{R}^3$ to its orthogonal projection $P(v) = (x, y, 0)$ in the xy -plane, determine the coordinate matrix $[P]_{\mathcal{B}}$ with respect to the basis

$$\mathcal{B} = \left\{ \boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

10. Consider the same problem given in Exercise 9, but use different bases – say,

$$\mathcal{B} = \left\{ \boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ and}$$

$$\mathcal{B}' = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

For the projector defined by $P(x, y, z) = (x, y, 0)$, determine $[P]_{\mathcal{B}\mathcal{B}'}$.

Action as Matrix Multiplication Let $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and let $\mathcal{B}$ and B' be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. For each $u \in \mathcal{U}$, the action of T on u is given by matrix multiplication between their coordinates in the sense that $[T(\boldsymbol{u})]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'}[\boldsymbol{u}]_{\mathcal{B}}$

11. Show how the action of the operator $D\Big(p(t)\Big) = \mathrm{d}p/\,\mathrm{d}t$ on the space $\mathcal{P}_3$ of polynomials of degree three or less is given by matrix multiplication.

Connections with Matrix Algebra

- If $T, L \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and if $\mathcal{B}$ and $\mathcal{B}'$ are bases for $\mathcal{U}$ and $\mathcal{V}$, then
  - ▷ $[\alpha T]_{\mathcal{B}\mathcal{B}'} = \alpha[T]_{\mathcal{B}\mathcal{B}'}$ for scalars α,
  - ▷ $[T + L]_{\mathcal{B}\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'} + [L]_{\mathcal{B}\mathcal{B}'}$.
- If $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $L \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, and if $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$ are bases for $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, respectively, then $LT \in \mathcal{L}(\mathcal{U}, \mathcal{W})$, and
  - ▷ $[LT]_{\mathcal{B}\mathcal{B}''} = [L]_{\mathcal{B}'\mathcal{B}''}[T]_{\mathcal{B}\mathcal{B}'}$.
- If $T \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ is invertible in the sense that $TT^{-1} = T^{-1}T = I$ for some $T^{-1} \in \mathcal{L}(\mathcal{U}, \mathcal{U})$, then for every basis $\mathcal{B}$ of $\mathcal{U}$,
  - ▷ $[T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}$.

12. Form the composition $C = LT$ of the two linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x + y, y - z) \text{ and } L(u, v) = (2u - v, u),$$

and then verify $[LT]_{\mathcal{B}\mathcal{B}''} = [L]_{\mathcal{B}'\mathcal{B}''}[T]_{\mathcal{B}\mathcal{B}'}$ and $[T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}$ (from the claim above) using the standard bases $\mathcal{S}_2$ and $\mathcal{S}_3$ for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

13. For the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, -2x + 4y)$, determine $[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.

14. For $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, let T be the linear operator on $\mathbb{R}^n$ defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$. That is, T is the operator defined by matrix multiplication. With respect to the standard basis $\mathcal{S}$, show that $[T]_{\mathcal{S}} = A$.

15. If T is a linear operator on a space $\mathcal{V}$ with basis $\mathcal{B}$, explain why $[T^k]_{\mathcal{B}} = [T]_{\mathcal{B}}^k$ for all nonnegative integers k.

16. **(IMC 2011.)** For some $A \in \mathrm{Mat}_{3\times3}(\mathbb{R})$ let λ_1, λ_2 and λ_3 denote three different real numbers such that

$$p(x) = \det(A - xI) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

For $i = 1, 2, 3$ let V_i denote spaces $V_i := \ker(A - \lambda_i I)$. Show that (a) $\dim(V_i) = 1$. (b) $\mathbb{R}^3 = V_1 + V_2 + V_3$. (c) $\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3$. (d) Discuss and carefully explain is it possible that for A we have both $A^2 + A^\top = I$ and $\mathrm{trace}(A) = 0$.

InC: 1, 6, 7, 9, 10, 11, 12. HW: 16 + several problems from the web page `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.